Quantum Logic: A Brief Introduction

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December 25th, 2018

Outline

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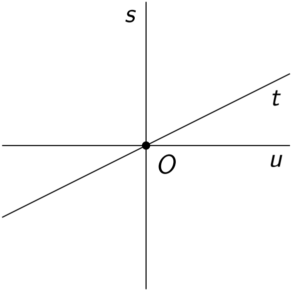
A Toy Model

Fix a point O in the three-dimensional Euclidean space E^3.

$\mathbf{L}$: the set of all lines in E^3 passing through O

$\not\perp$: the binary non-perpendicularity relation on $\mathbf{L}$

for any $s, t \in \mathbf{L}$, $s \not\perp t$, iff s and t are not perpendicular



Orthocomplement

For any $P \subseteq \mathbf{L}$, its orthocomplement is defined as follows:

$$\begin{aligned} \sim P &\overset{\text{def}}{=} \{s \in \mathbf{L} \mid s \not\perp t \Rightarrow t \notin P, \text{ for any } t \in \Sigma\} \\ &= \{s \in \mathbf{L} \mid s \text{ is perpendicular to all } u \in P\} \end{aligned}$$

Example

1. For $P = \emptyset$, $\sim P = \mathbf{L}$.
2. For P containing exactly one line $s \in \mathbf{L}$, $\sim P$ is the plane perpendicular to s.
3. For P containing two different lines which determine a plane Q with $P \subseteq Q$, $\sim P$ only contains the line perpendicular to Q.
4. For P containing three lines which are not on the same plane, $\sim P = \emptyset$.

Bi-Orthogonally Closed Set

$P \subseteq \mathbf{L}$ is bi-orthogonally closed, if $P = \sim\sim P$

Fact

In this example, there are four kinds of bi-orthogonally closed sets:

1. $\emptyset$
2. singletons
3. planes
4. $\mathbf{L}$

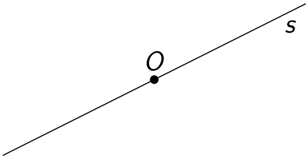
Some Properties of Non-Perpendicularity

1. Reflexivity
2. Symmetry
3. Separation
4. Superposition
5. Representation

(1) Reflexivity

Reflexivity

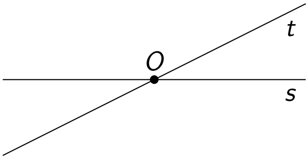
$s \not\perp s$, for every $s \in \mathbf{L}$.



(2) Symmetry

Symmetry

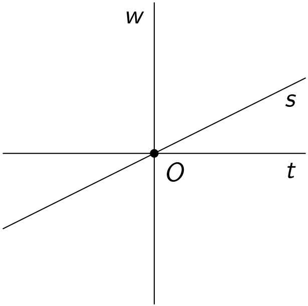
$s \not\perp t \Rightarrow t \not\perp s$, for any $s, t \in \mathbf{L}$.



(3) Separation

Separation

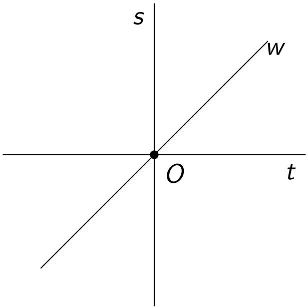
For any $s, t \in \mathbf{L}$ satisfying $s \neq t$, there is a $w \in \mathbf{L}$ such that $w \not\perp s$ but not $w \not\perp t$.



(4) Superposition

Superposition

For any $s, t \in \mathbf{L}$, there is a $w \in \mathbf{L}$ such that $w \not\perp s$ and $w \not\perp t$.



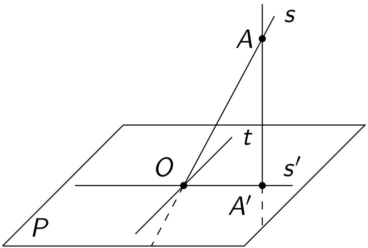
(5) Representation

Definition (Representative)

For any $s \in \mathbf{L}$ and $P \subseteq \mathbf{L}$, $s' \in \mathbf{L}$ is a representative of s in P, if $s' \in P$ and, for each $t \in P$, $s \not\perp t \Leftrightarrow s' \not\perp t$.

Representation

For any $P \subseteq \mathbf{L}$ and $s \in \mathbf{L}$ such that $P = \sim\sim P$ and $s \notin \sim P$, s has a representative in P.



Some Properties of Non-Perpendicularity (Summary)

1. Reflexivity
   $s \not\perp t \Rightarrow t \not\perp s$, for any $s, t \in \mathbf{L}$
2. Symmetry
   $s \not\perp t \Rightarrow t \not\perp s$, for any $s, t \in \mathbf{L}$
3. Separation
   For any $s, t \in \mathbf{L}$ satisfying $s \neq t$, there is a $w \in \mathbf{L}$ such that $w \not\perp s$ but not $w \not\perp t$
4. Superposition
   For any $s, t \in \mathbf{L}$, there is a $w \in \mathbf{L}$ such that $w \not\perp s$ and $w \not\perp t$.
5. Existence of Representative
   For any $P \subseteq \mathbf{L}$ and $s \in \mathbf{L}$ such that $P = {\sim}{\sim}P$ and $s \notin {\sim}P$, s has a representative in P

Quantum Kripke Frame

Definition (Kripke Frame)

A Kripke frame $\mathfrak{F}$ is a tuple $(\Sigma, \to)$, where $\Sigma \neq \emptyset$ and $\to \subseteq \Sigma \times \Sigma$.

Definition (Quantum Kripke Frame)

A quantum Kripke frame is a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying:

1. Reflexivity: $s \to s$, for each $s \in \Sigma$.
2. Symmetry: $s \not\to t$ implies $t \not\to s$, for any $s, t \in \Sigma$.
3. Separation:
   For any $s, t \in \Sigma$, if $s \neq t$, then there is a $w \in \Sigma$ such that $w \not\to s$ and $w \to t$.
4. Superposition:
   For any $s, t \in \Sigma$, there is a $w \in \Sigma$ such that $w \to s$ and $w \to t$.
5. Representation:
   For any $s \in \Sigma$ and $P \subseteq \Sigma$, if $\sim\sim P = P$ and $s \notin \sim P$, then there is an $s_{\|} \in P$ such that $s \not\to w \;\Leftrightarrow\; s_{\|} \not\to w$ holds for each $w \in P$.

Orthocomplement and Bi-orthogonally Closed Subset

Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame.

Definition (Orthocomplement)

For a $P \subseteq \Sigma$, the orthocomplement of P is defined as follows:

$$\sim P \stackrel{\text{def}}{=} \{s \in \Sigma \mid s \to t \Rightarrow t \notin P \text{ holds for each } t \in \Sigma\}$$

Definition (Bi-orthogonally Closed Subset)

$P \subseteq \Sigma$ is bi-orthogonally closed, if $P = \sim\sim P$.

$\mathcal{L}_{\mathfrak{F}}$: the set of all bi-orthogonally closed subsets of $\mathfrak{F}$.

Simple Facts about Orthocomplements

Let $\mathfrak{F} = (\Sigma, \rightarrow)$ be a Kripke frame satisfying Reflexivity and Symmetry.

1. $\sim\emptyset = \Sigma$ and $\sim\Sigma = \emptyset$.
2. $P \subseteq \sim\sim P$, for each $P \subseteq \Sigma$.
3. $P \subseteq Q$ implies that $\sim Q \subseteq \sim P$, for any $P, Q \subseteq \Sigma$.
4. $\sim P \in \mathcal{L}_{\mathfrak{F}}$, for each $P \subseteq \Sigma$.
5. $P \cap Q \in \mathcal{L}_{\mathfrak{F}}$, for any $P, Q \in \mathcal{L}_{\mathfrak{F}}$.

Why Is This Called Quantum? A Lite Math Explanation

- $(\mathbf{L}, \not\perp)$ is a quantum Kripke frame and is abstracted from E^3.
- According to analytic geometry, E^3 is the same as $\mathbb{R}^3$.
- Generalizing the above to arbitrary finite dimensions, we get $\mathbb{R}^n$.
  The math theory of them is linear algebra on the real numbers.
- Generalizing the above to $\mathbb{C}$, we get $\mathbb{C}^n$.
  The math theory of them is linear algebra on the complex numbers.
  This is the math of quantum computation and quantum information.
- Generalizing the above to infinite dimensions, we get Hilbert spaces over $\mathbb{C}$.
  The math theory of them is functional analysis on the complex numbers.
  This is the math of quantum physics.
- From each Hilbert space over $\mathbb{C}$, we can extract a quantum Kripke frame.

Why Is This Called Quantum? A Lite Phys. Explanation

- A quantum system is described by a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$.
- A (pure) state of the system is described by an element in Σ.
- For $s, t \in \Sigma$, $s \to t$ means that s and t can not be perfectly discriminated.
- A property of the system is described by a bi-orthogonally subset of Σ.

Why Do Bi-orthogonally Closed Sets Describe Properties?

Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame satisfying Reflexivity and Symmetry.

Definition (Opposite Pair and Maximal Opposite Pair)

An opposite pair in $\mathfrak{F}$ is a tuple (P, Q) where $P \subseteq \Sigma$, $Q \subseteq \Sigma$ and $s \not\to t$ for any $s \in P$ and $t \in Q$.
An opposite pair (P, Q) in $\mathfrak{F}$ is maximal, if, for each opposite pair (P', Q') in $\mathfrak{F}$, $P \subseteq P'$ and $Q \subseteq Q'$ imply that $P = P'$ and $Q = Q'$.

Proposition

For each maximal opposite pair (P, Q) in $\mathfrak{F}$, both P and Q are bi-orthogonally closed.

Proposition

For each $P \subseteq \Sigma$, the following are equivalent:

(a) P is bi-orthogonally closed;

(b) $(P, \sim P)$ is a maximal opposite pair in $\mathfrak{F}$.

Quantum Test and Maximal Opposite Pair

Consider a quantum system described by a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$.

Tests of this quantum system are described by maximal opposite pairs of $\mathfrak{F}$.

Assume that it is in the state $s \in \Sigma$, and we do a test described by (P_0, P_1):

1. if $s \in P_0$, then the outcome will be 0 and the state after the test is s;
2. if $s \in P_1$, then the outcome will be 1 and the state after the test is s;
3. if $s \notin P_0 \cup P_1$, then there are two possibilities:
    1. the outcome is 0, and the state after the test is the representative of s in P_0;
    2. the outcome is 1, and the state after the test is the representative of s in P_1.

Formal Languages

Let PV be a set of propositional variables.

Definition (Propositional Formula)

The notion of a (propositional) formula is defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi, \quad p \in PV$$

Form: the set of (propositional) formulas

Definition (Modal Formula)

The notion of a modal formula is defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi, \quad p \in PV$$

$Form_M$: the set of modal formulas

Outline

Partially Ordered Set

Definition (Partially Ordered Set)

A partially ordered set is a tuple $\mathfrak{P} = (P, \leq)$, where $P \neq \emptyset$ and $\leq \subseteq P \times P$ such that, for any $a, b, c \in P$,

1. $a \leq a$;
2. $a \leq b$ and $b \leq c$ imply that $a \leq c$;
3. $a \leq b$ and $b \leq a$ imply that $a = b$.

Fact

1. For each set A, $(\wp(A), \subseteq)$ is a partially ordered set.
2. For each quantum Kripke frame $\mathfrak{F}$, $(\mathcal{L}(\mathfrak{F}), \subseteq)$ is a partially ordered set.

Lattice

Definition (Lattice)

A lattice is a partially order set $\mathfrak{L} = (L, \leq)$ where any pair of elements $a, b \in L$ has an infimum (called meet) $a \wedge b$ and a supremum (called join) $a \vee b$.

Fact

1. For each set A, $(\wp(A), \subseteq)$ is a lattice with $\cap$ as the meet and $\cup$ as the join.
2. For each quantum Kripke frame $\mathfrak{F}$, $(\mathcal{L}_{\mathfrak{F}}, \subseteq)$ is a lattice with $P \cap Q$ as the meet and $P \sqcup Q = \sim(\sim P \cap \sim Q)$ as the join for any $P, Q \in \mathcal{L}_{\mathfrak{F}}$.

Bounded Lattice

Definition (Bounded Lattice)

A bounded lattice is a tuple $\mathfrak{L} = (L, \leq, O, I)$ where $(L, \leq)$ is a lattice and $O, I \in L$ satisfy that $O \leq a \leq I$ holds for each $a \in L$.

Fact

1. For each set A, $(\wp(A), \subseteq, \emptyset, \Sigma)$ is a bounded lattice.
2. For each quantum Kripke frame $\mathfrak{F}$, $(\mathcal{L}(\mathfrak{F}), \subseteq, \emptyset, \Sigma)$ is a bounded lattice.

(Lattice-theoretic) Orthocomplement

Definition (Orthocomplementation)

An orthocomplementation on a bounded lattice $\mathfrak{L} = (L, \leq, O, I)$ is a function $(\cdot)' : L \to L$ such that, for any $a, b \in L$,

1. $a \wedge a' = O$ and $a \vee a' = I$;
2. $a \leq b$ implies that $b' \leq a'$;
3. $(a')' = a$.

For each $a \in L$, a' is called the (lattice-theoretic) orthocomplement of a. A tuple $\mathfrak{L} = (L, \leq, (\cdot)', O, I)$ is an ortho-lattice, if $(L, \leq, O, I)$ is a bounded lattice and $(\cdot)'$ is an orthocomplementation on $(L, \leq, O, I)$.

Fact

1. For each set A, set-theoretic complement $A \setminus \cdot$ is an orthocomplementation on the bounded lattice $(\wp(A), \subseteq, \emptyset, \Sigma)$.
2. For each quantum Kripke frame $\mathfrak{F}$, orthocomplement $\sim(\cdot)$ is an orthocomplementation on the bounded lattice $(\mathcal{L}(\mathfrak{F}), \subseteq, \emptyset, \Sigma)$.

De Morgen's Law

Proposition (De Morgen's Law)

Let $\mathfrak{L} = (L, \leq, (\cdot)', O, I)$ be an ortho-lattice.
For any $a, b \in L$,

$$(a \wedge b)' = a' \vee b'$$
$$(a \vee b)' = a' \wedge b'$$

Proof.

Since $a \wedge b \leq a$, $a' \leq (a \wedge b)'$.
Since $a \wedge b \leq b$, $b' \leq (a \wedge b)'$.
Therefore, $a' \vee b' \leq (a \wedge b)'$.

Since $a' \leq a' \vee b'$, $(a' \vee b')' \leq a'' = a$.
Since $b' \leq a' \vee b'$, $(a' \vee b')' \leq b'' = b$.
Therefore, $(a' \vee b')' \leq a \wedge b$.
It follows that $(a \wedge b)' \leq (a' \vee b')'' = a' \vee b'$. □

Distributivity

Definition (Distributive Lattice)

A lattice $\mathfrak{L} = (L, \leq)$ is a distributive lattice, if for each $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

In fact, in a lattice, each one of them implies the other.

Fact

For each set A, $(\wp(A), \subseteq)$ is a distributive lattice.

Definition (Boolean Algebra)

A Boolean algebra is a distributive ortho-lattice.

Fact

For each set A, $(\wp(A), \subseteq, A \setminus (\cdot), \emptyset, A)$ is a Boolean algebra.

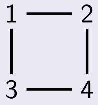
Non-distributivity

Proposition

There is a quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ such that $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(\cdot), \emptyset, \Sigma)$ is not a distributive lattice, and thus not a Boolean algebra.

Proof.

Consider the following Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$:



$$\mathcal{L}_{\mathfrak{F}} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \Sigma\}$$

$$(\{1\} \cap \{2\}) \sqcup \{3\} = \emptyset \sqcup \{3\} = \{3\} \neq \Sigma = \Sigma \cap \Sigma = (\{1\} \sqcup \{3\}) \cap (\{2\} \sqcup \{3\})$$

$$(\{1\} \sqcup \{2\}) \cap \{3\} = \Sigma \cap \{3\} = \{3\} \neq \emptyset = \emptyset \sqcup \emptyset = (\{1\} \cap \{3\}) \sqcup (\{2\} \cap \{3\})$$

□

Orthomodularity

Theorem

For each quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, the following holds:

$$P \cap (\sim P \sqcup (P \cap Q)) \subseteq Q, \text{ for any } P, Q \in \mathcal{L}_{\mathfrak{F}}$$

Proof.

Assume that $s \in P$ and $s \in \sim P \sqcup (P \cap Q)$.
Let t be arbitrary such that $s \rightarrow t$.
By Symmetry $t \rightarrow s$. Since $s \in P$, $t \notin \sim P$.
By Representation there is a $t' \in P$ such that, for each $u \in P$,
$t \rightarrow u \Leftrightarrow t' \rightarrow u$.
Since $s \in P$ and $t \rightarrow s$, $t' \rightarrow s$. By Symmetry $s \rightarrow t'$.
Since $s \in \sim P \sqcup (P \cap Q)$, $t' \notin P \cap \sim(P \cap Q)$.
Since $t' \in P$, $t' \notin \sim(P \cap Q)$.
Hence there is a $w \in P \cap Q$ such that $t' \rightarrow w$.
Since $w \in P$ and $t' \rightarrow w$, $t \rightarrow w$.
Since $w \in Q$, $t \notin \sim Q$.
Therefore, $s \in \sim\sim Q = Q$. □

Orthomodular Lattice

Definition (Orthomodular Lattice)

An orthomodular lattice is an ortho-lattice $\mathfrak{L} = (L, \leq, (\cdot)', O, I)$ satisfying the following orthomodular law, i.e.

$$a \wedge (a' \vee (a \wedge b)) \leq b, \text{ for any } a, b \in L$$

Lemma [Mittelstaedt, 1978]

In an ortho-lattice $\mathfrak{L} = (L, \leq, (\cdot)', O, I)$, the following are equivalent:

(i) $a \wedge (a' \vee (a \wedge b)) \leq b$, for any $a, b \in L$;

(ii) $a \leq b$ implies $a = b \wedge (a \vee b')$, for any $a, b \in L$;

(iii) $a \leq b$ implies $b = a \vee (a' \wedge b)$, for any $a, b \in L$;

(iv) $a \leq b$ and $c \leq b'$ imply $b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c)$, for any $a, b, c \in L$.

Examples of Orthomodular Lattices

Fact

1. Every Boolean algebra is an orthomodular lattice.
2. For each quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(\cdot), \emptyset, \Sigma)$ is an orthomodular lattice.

A Famous Open Problem

Open Problem

Can every orthomodular lattice be embedded into a complete orthomodular lattice?

Outline

Boolean Algebras and Classical Logic

Definition (Assignment on a Boolean Algebra)

An assignment σ on a Boolean algebra $\mathfrak{L} = (L, \leq, (\cdot)', O, I)$ is a function from *Form* to L such that

1. $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$;
2. $\sigma(\neg\varphi) = (\sigma(\varphi))'$.

Definition (Semantic Consequence w.r.t **BA**)

For each $\Gamma \subseteq Form$ and $\phi \in Form$, $\Gamma \Vdash_{\mathbf{BA}} \phi$,
if, for each Boolean algebra $\mathfrak{L}$, assignment σ on $\mathfrak{L}$ and each $a \in L$,
$a \leq \sigma(\psi)$ for all $\psi \in \Gamma$ implies that $a \leq \sigma(\phi)$.

Theorem

For each $\Gamma \subseteq Form$ and $\phi \in Form$,

$$\Gamma \vdash_{PC} \varphi \;\Leftrightarrow\; \Gamma \Vdash_{\mathbf{BA}} \varphi$$

Ortho-lattices and Semantic Consequence

Definition (Assignment on an Ortho-lattice)

An assignment σ on an ortho-lattice $\mathfrak{L} = (L, \leq, (\cdot)', O, I)$ is a function from *Form* to L such that

1. $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$;
2. $\sigma(\neg\varphi) = (\sigma(\varphi))'$.

Definition (Semantic Consequence w.r.t a Class of Ortho-Lattice)

Let **C** be a subclass of the class of ortho-lattices.
For each $\Gamma \subseteq Form$ and $\phi \in Form$, $\Gamma \Vdash_{\mathbf{C}} \phi$,
if, for each ortho-lattice $\mathfrak{L} \in \mathbf{C}$, assignment σ on $\mathfrak{L}$ and each $a \in L$,
$a \leq \sigma(\psi)$ for all $\psi \in \Gamma$ implies that $a \leq \sigma(\phi)$.

Definition

OL: the class of all ortho-lattices
OML: the class of all orthomodular lattices

An Axiomatization of Ortho-Logic

The first axiomatization of ortho-logic is given in [Goldblatt, 1974], and the following one is from [Chiara and Giuntini, 2002].

Definition (Ortho-Logic)

$$\frac{}{\Gamma \cup \{\varphi\} \vdash \varphi} \qquad \frac{}{\Gamma \cup \{\varphi \wedge \psi\} \vdash \varphi} \qquad \frac{}{\Gamma \cup \{\varphi \wedge \psi\} \vdash \psi}$$

$$\frac{}{\Gamma \cup \{\varphi\} \vdash \neg\neg\varphi} \qquad \frac{}{\Gamma \cup \{\neg\neg\varphi\} \vdash \varphi} \qquad \frac{}{\Gamma \cup \{\varphi \wedge \neg\varphi\} \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \cup \{\varphi\} \vdash \psi}{\Gamma \cup \Delta \vdash \psi} \qquad \frac{\Gamma \cup \{\varphi, \psi\} \vdash \theta}{\Gamma \cup \{\varphi \wedge \psi\} \vdash \theta} \qquad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}$$

$$\frac{\{\varphi\} \vdash \psi \qquad \{\varphi\} \vdash \neg\psi}{\vdash \neg\varphi} \qquad \frac{\{\varphi\} \vdash \psi}{\{\neg\psi\} \vdash \neg\varphi}$$

Derivation and Syntactic Consequence

Definition (Sequent)

$\Gamma \vdash \varphi$, where $\Gamma \subseteq Form$ and $\varphi \in Form$, is called a sequent.

Definition (Derivation)

A derivation is a *finite* sequence of sequents, each of which satisfies one of the following:

- it is the conclusion of an improper rule;
- it is the conclusion of a proper rule whose premises are previous elements in this sequence.

Definition (Syntactic Consequence)

$\varphi \in Form$ is a syntactic consequence of $\Gamma \subseteq Form$ in ortho-logic ($\Gamma \vdash_{OL} \varphi$), if there is a derivation such that $\Gamma \vdash \varphi$ is the last element.

An Axiomatization of Orthomodular Logic

Definition (Orthomodular Logic)

Orthomodular logic is that of ortho-logic plus the following improper rule:

$$\frac{}{\varphi \wedge \neg(\varphi \wedge \neg(\varphi \wedge \psi)) \vdash \psi}$$

The notions of derivation and syntactic consequence ($\Gamma \vdash_{OML} \phi$) can be defined similar to those for ortho-logic.

Characterization Theorems

Theorem

For each $\Gamma \subseteq Form$ and $\phi \in Form$,

$$\Gamma \vdash_{OL} \varphi \;\Leftrightarrow\; \Gamma \Vdash_{\mathbf{OL}} \varphi$$

Theorem

For each $\Gamma \subseteq Form$ and $\phi \in Form$,

$$\Gamma \vdash_{OML} \varphi \;\Leftrightarrow\; \Gamma \Vdash_{\mathbf{OML}} \varphi$$

Outline

Compatible Elements

Definition (Compatible Elements)

In an ortho-lattice $\mathfrak{L} = (L, \leq, (\cdot)' O, I)$, $a, b \in L$ is compatible, denoted by $a \sim b$, if

$$a = (a \wedge b) \vee (a \wedge b')$$

Theorem [Mittelstaedt, 1978]

In an ortho-lattice $\mathfrak{L}$, the following are equivalent:

(i) the compatibility relation $\sim$ is symmetric;

(ii) orthomodularity holds, i.e. $\mathfrak{L}$ is an orthomodular lattice.

Properties of Compatible Elements

Theorem [Mittelstaedt, 1978]

In an orthomodular lattice $\mathfrak{L} = (L, \leq, (\cdot)')$,

1. $a \leq b$ implies that $a \sim b$;
2. $b \sim a$ and $c \sim a$ imply that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;
3. the relation $\sim$ is closed under $(\cdot)'$, $\vee$ and $\wedge$, i.e.
    1. $a \sim b$ implies that $a \sim b'$;
    2. $a \sim b$ and $a \sim c$ imply that $a \sim (b \vee c)$;
    3. $a \sim b$ and $a \sim c$ imply that $a \sim (b \wedge c)$.

Corollary [Mittelstaedt, 1978]

K1 $\sim$ is symmetric;

K2 $\leq \subseteq \sim$;

K3 If $A \subseteq L$ satisfies $A \times A \subseteq \sim$, A generates a Boolean sub-lattice of $\mathfrak{L}$;

K4 If $A \subseteq L$ forms a Boolean sub-lattice of $\mathfrak{L}$, $A \times A \subseteq \sim$.

Characterization of Compatibility

Theorem [Mittelstaedt, 1978]

In an orthomodular lattice, every binary relation satisfying (K1) - (K4) is equal to $\sim$.

Theorem [Mittelstaedt, 1978]

In an ortho-lattice $\mathfrak{L}$, the following are equivalent:

(i) Orthomodularity holds, i.e. $\mathfrak{L}$ is an orthomodular lattice;

(ii) there exists a binary relation on $\mathfrak{L}$ satisfying (K1) - (K4).

Indicator of Compatibility

Theorem [Mittelstaedt, 1978]

In an orthomodular lattice, for any two elements a and b,

$$a \sim b \Leftrightarrow k(a, b) = I$$

where

$$k(a, b) = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b')$$

Direct Product and Reducibility

Definition (Direct Product of Ortho-Lattice)

Given two ortho-lattices $\mathfrak{L}_1 = (L_1, \leq_1, (\cdot)^{\perp_1}, O_1, I_1)$ and $\mathfrak{L}_2 = (L_2, \leq_2, (\cdot)^{\perp_2}, O_2, I_2)$, the direct product of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is a tuple $(L, \leq, (\cdot)')$ such that:

1. $L = L_1 \times L_2$;
2. for any $(a_1, a_2), (b_1, b_2) \in L$, $(a_1, a_2) \leq (b_1, b_2)$, if $a_1 \leq_1 b_1$ and $a_2 \leq_2 b_2$;
3. for any $(a_1, a_2) \in L$, $(a_1, a_2)' = (a_1^{\perp_1}, a_2^{\perp_2})$.

Definition (Reducibility)

An ortho-lattice is reducible, if it is isomorphic to the direct product of two non-trivial ortho-lattices.
Otherwise, it is irreducible.

Compatibility and Reducibility

Theorem [Piron, 1976]

In an orthomodular lattice $\mathfrak{L} = (L, \leq, (\cdot)', O, I)$, if there is a $b \in L$ which is compatible with every element of L, then $\mathfrak{L}$ is reducible.
In particular, it is isomorphic to the direct product $[O, b] \times [O, b']$ via the map $\theta :: a \mapsto (a \wedge b, a \wedge b')$.

Corollary

Every Boolean algebra with more than 2 elements is reducible.

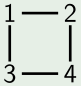
Outline

The Implication Problem

A Requirement for Implication

$$a \to b = I \quad \Leftrightarrow \quad a \leq b$$

Material Implication Fails



$\sim\{1\} \sqcup \{2\} = \{4\} \sqcup \{2\} = \{1, 2, 3, 4\}$ but $\{1\} \not\subseteq \{2\}$.

Theorem

In an ortho-lattice $\mathfrak{L}$, if, for any two elements $a, b \in L$, there is an $a \to b \in L$ such that

$$c \wedge a \leq b \Leftrightarrow c \leq a \to b, \text{ for each } c \in L,$$

then $\mathfrak{L}$ is distributive and thus is a Boolean algebra.

The Search of an Implication

Theorem [Kalmbach, 1983]

In an orthomodular lattice freely generated by two elements there are only five polynomial binary operations $\rightarrow$ satisfying the condition $a \leq b$ if and only if $a \rightarrow b = I$:

1. $a \rightarrow_1 b = a' \vee (a \wedge b)$;
2. $a \rightarrow_2 b = b \vee (a' \wedge b')$;
3. $a \rightarrow_3 b = (a' \wedge b) \vee (a \wedge b) \vee (a' \wedge b')$;
4. $a \rightarrow_4 b = (a' \wedge b) \vee (a \wedge b) \vee ((a' \vee b) \wedge b')$;
5. $a \rightarrow_5 b = (a' \wedge b) \vee (a' \wedge b') \vee (a \wedge (a' \vee b))$.

Proposition [Kotas, 1967]

In an orthomodular lattice, $i = 1$, if and only if $\rightarrow_i$ has the following property:

$$a \sim b \text{ implies that } c \wedge a \leq b \Leftrightarrow c \leq a \rightarrow_i b \text{ for each } c \in L.$$

Sasaki Hook

Definition (Sasaki Hook)

In an ortho-lattice, the Sasaki hook of a and b is the element:

$$a \xrightarrow{S} b \overset{\text{def}}{=} a' \vee (a \wedge b)$$

Theorem [Mittelstaedt, 1978]

In an ortho-lattice $\mathfrak{L}$, the following are equivalent:

(i) $\mathfrak{L}$ satisfies orthomodularity, i.e. is an orthomodular lattice;

(ii) for any a and b, there is an element $a \xrightarrow{S} b$ satisfying:

1. $a \wedge (a \xrightarrow{S} b) \leq b$;
2. $a \wedge c \leq b \Rightarrow a' \vee (a \wedge c) \leq a \xrightarrow{S} b$.

When one (and thus both) of these conditions holds, $a \xrightarrow{S} b = a' \vee (a \wedge b)$ for any a and b.

Properties of the Sasaki Hook

Theorem [Mittelstaedt, 1978]

The following hold in all orthomodular lattices:

1. $a \vee (a \xrightarrow{S} b) = I$
2. $((a \xrightarrow{S} b) \xrightarrow{S} a) \xrightarrow{S} a = I$ (Peirce's Law)

Fact

The following does **NOT** hold in general in orthomodular lattices:

- $a \xrightarrow{S} (b \xrightarrow{S} a) = I$
- $(a \xrightarrow{S} b \xrightarrow{S} c) \xrightarrow{S} (a \xrightarrow{S} b) \xrightarrow{S} a \xrightarrow{S} c = I$

A Counterfactual Reading of the Sasaki Hook

Consider a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, $s \in \Sigma$ and $P, Q \in \mathcal{L}_{\mathfrak{F}}$.

Fact

The following are equivalent:

(i) $s \in P \xrightarrow{S} Q$;

(ii) for each representative s' of s in P, $s' \in Q$.

Define a function
$F : \mathcal{L}_{\mathfrak{F}} \times \Sigma \to \Sigma :: (P, s) \mapsto \{s' \in \Sigma \mid s' \text{ is a representative of } s \text{ in } P\}$

$$s \subseteq P \xrightarrow{S} Q \iff F(P, s) \subseteq W$$

The system in a state has property $P \xrightarrow{S} Q$, if the system has property Q after a test of the property P yielding a positive result.

Outline

Outline

Ortho-frame and Ortho-model

Definition (Ortho-frame)

An ortho-frame is a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying Reflexivity and Symmetry.

Definition (Ortho-model)

An ortho-model is a tuple $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is an ortho-frame and $V : PV \to \mathcal{L}_{\mathfrak{F}}$ is a function.

Truth and Semantic Consequence

Definition (Truth)

$\varphi \in Form$ being true at a point $s \in \Sigma$ in an ortho-model $\mathfrak{M} = ((\Sigma, \perp), V)$, $\mathfrak{M}, s \Vdash \varphi$, is defined recursively as follows:

$$\begin{aligned} \mathfrak{M}, s \Vdash p &\Leftrightarrow s \in V(p) \\ \mathfrak{M}, s \Vdash \varphi \wedge \psi &\Leftrightarrow \mathfrak{M}, s \Vdash \varphi \text{ and } \mathfrak{M}, s \Vdash \psi \\ \mathfrak{M}, s \Vdash \neg\varphi &\Leftrightarrow s \to t \text{ implies that } \mathfrak{M}, t \nVdash \varphi, \text{ for all } t \in \Sigma \end{aligned}$$

Definition (Semantic Consequence)

$\varphi \in Form$ is a semantic consequence of $\Gamma \subseteq Form$, denoted as $\Gamma \Vdash_{\mathbf{OF}} \varphi$, if $\mathfrak{M}, s \Vdash \Gamma$ implies that $\mathfrak{M}, s \Vdash \varphi$, for every ortho-model $\mathfrak{M}$ and s in the underlying set of $\mathfrak{M}$.

Characterization Theorem

Theorem

For each $\Gamma \subseteq \mathit{Form}$ and $\phi \in \mathit{Form}$,

$$\Gamma \vdash_{OL} \varphi \Leftrightarrow \Gamma \Vdash_{\mathbf{OF}} \varphi$$

Open Problem [Goldblatt, 1974]

What special kind of ortho-frames does orthomodular logic axiomatize?

Translation into the Modal Logic **KTB**

A translation map $T : \mathit{Form} \to \mathit{Form}_M$ can be defined as follows:

$$
\begin{aligned}
T(p) &= \Box\neg\Box\neg p \\
T(\varphi \wedge \psi) &= T(\varphi) \wedge T(\psi) \\
T(\neg\varphi) &= \Box\neg T(\varphi)
\end{aligned}
$$

Theorem [Goldblatt, 1974]

For any $\Gamma \subseteq \mathit{Form}$ and $\varphi \in \mathit{Form}$,

$$\Gamma \vdash_{OL} \varphi \;\Leftrightarrow\; \{T(\psi) \mid \psi \in \Gamma\} \vdash_{\mathbf{KTB}} T(\varphi).$$

Intuitionistic Logic

Please note that the minimal set of primitive connectives in intuitionistic logic includes $\bot$, $\wedge$, $\vee$, $\rightarrow$.

Definition (Int-frame)

An int-frame is a Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ satisfying Reflexivity and Transitivity.

Definition (Int-model)

An int-model is a tuple $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F} = (\Sigma, \rightarrow)$ is an int-frame and V is a function from PV to the set of all persistent/upward closed subsets of $\mathfrak{F}$, i.e. sets $P \subseteq \Sigma$ satisfying:

$$\text{for each } s, t \in \Sigma, \text{ if } s \in P \text{ and } s \rightarrow t, \text{ then } t \in P.$$

Translation From Intuitionistic Logic into the **S4**

The Tarski-Mckinsey translation T can be defined as follows:

$$\begin{aligned} T(p) &= \Box p \\ T(\bot) &= \bot \\ T(\varphi \wedge \psi) &= T(\varphi) \wedge T(\psi) \\ T(\varphi \vee \psi) &= T(\varphi) \vee T(\psi) \\ T(\varphi \rightarrow \psi) &= \Box(T(\varphi) \rightarrow T(\psi)) \end{aligned}$$

Theorem

For any set of formulas Γ and formula φ in the propositional language of intuitionistic logic,

$$\Gamma \vdash_{Int} \varphi \;\Leftrightarrow\; \{T(\psi) \mid \psi \in \Gamma\} \vdash_{\mathbf{S4}} T(\varphi).$$

A General Relational Semantics for Propositional Logic

In fact, the relational semantics of ortho-logic and that of the $\{\neg, \wedge\}$-fragment of intuitionistic logic can be unified under a general relational semantics for propositional logic.

Proposition

Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame.

Definition (Proposition)

A proposition on $\mathfrak{F}$ is a set $P \subseteq \Sigma$ such that, for each $s \in \Sigma$, the following are equivalent:

(i) $s \in P$;

(ii) for any $t \in \Sigma$, if $s \to t$, there is a $u \in \Sigma$ satisfying $u \in P$ and $u \to t$.

For each $P \subseteq \Sigma$, the direction from (i) to (ii) always holds, but the conserve may not.

Facts about Propositions

Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame.

Lemma

1. Σ is a proposition on $\mathfrak{F}$.
2. The set of all dead points is a proposition on $\mathfrak{F}$.

Lemma

For any propositions P and Q on $\mathfrak{F}$, $P \cap Q$ is a proposition.

Lemma

For each proposition P on $\mathfrak{F}$, $\sim P$ is a proposition.

Relational Semantics of Propositional Logic

Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame.

$\mathcal{P}_{\mathfrak{F}}$: the set of propositions on $\mathfrak{F}$

Definition (Model)

A model on $\mathfrak{F}$ is a tuple $\mathfrak{M} = (\mathfrak{F}, V)$, where $V : PV \to \mathcal{P}_{\mathfrak{F}}$ is a function.

Definition (Truth)

$\varphi \in Form$ being true at a point $s \in \Sigma$ in a model $\mathfrak{M} = ((\Sigma, \perp), V)$, $\mathfrak{M}, s \Vdash \varphi$, is defined recursively as follows:

$$\begin{aligned} \mathfrak{M}, s \Vdash p &\Leftrightarrow s \in V(p) \\ \mathfrak{M}, s \Vdash \varphi \wedge \psi &\Leftrightarrow \mathfrak{M}, s \Vdash \varphi \text{ and } \mathfrak{M}, s \Vdash \psi \\ \mathfrak{M}, s \Vdash \neg\varphi &\Leftrightarrow s \to t \text{ implies that } \mathfrak{M}, t \nVdash \varphi, \text{ for all } t \in \Sigma \end{aligned}$$

Special Case 1: Ortho-logic

Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame satisfying Reflexivity and Symmetry.

Proposition [Chiara and Giuntini, 2002]

For each $P \subseteq \Sigma$, the following is equivalent:

(a) $P \in \mathcal{P}_{\mathfrak{F}}$;

(b) P is bi-orthogonally closed, i.e. $P = \sim\sim P$.

$$\begin{aligned}
& P \in \mathcal{P}_{\mathfrak{F}} \\
\Leftrightarrow\ & \forall s[s \in P \text{ iff } \forall t(s \to t \Rightarrow \exists u(u \in P \text{ and } u \to t))] \\
\Leftrightarrow\ & \forall s[s \in P \text{ iff } \forall t(s \to t \Rightarrow \exists u(u \in P \text{ and } t \to u))] \qquad \text{(Symmetry)} \\
\Leftrightarrow\ & \forall s[s \in P \text{ iff } \forall t(\forall u(t \to u \to u \notin P) \Rightarrow s \not\to t)] \\
\Leftrightarrow\ & \forall s[s \in P \text{ iff } \forall t(t \in \sim P \Rightarrow s \not\to t)] \\
\Leftrightarrow\ & \forall s[s \in P \text{ iff } \forall t(s \to t \Rightarrow t \notin \sim P)] \\
\Leftrightarrow\ & \forall s[s \in P \text{ iff } s \in \sim\sim P] \\
\Leftrightarrow\ & P = \sim\sim P
\end{aligned}$$

Special Case 2: Intuitionistic Logic

Let $\mathfrak{F} = (\Sigma, \rightarrow)$ be a Kripke frame satisfying Reflexivity and Transitivity.

Proposition [Chiara and Giuntini, 2002]

For each $P \subseteq \Sigma$, the following is equivalent:

(a) $P \in \mathcal{P}_{\mathfrak{F}}$;

(b) P is persistent/upward closed.

From (b) to (a).

Suppose that (b) holds, i.e. P is persistent.
Let s be arbitrary.
It suffices to prove the direction from (ii) to (i).
Assume that $\forall t(s \rightarrow t \Rightarrow \exists u(u \in P \text{ and } u \rightarrow t))$.
By **Reflexivity** $s \rightarrow s$.
Hence there is a u such that $u \in P$ and $u \rightarrow s$.
Since P is persistent, $s \in P$. □

Intuitionistic Logic (2)

From (a) to (b).

Assume that (a) holds, i.e. $P \in \mathcal{P}_{\mathfrak{F}}$.
Let s, t be arbitrary such that $s \in P$ and $s \to t$.
By assumption there is a u such that $u \in P$ and $u \to t$.
Let v be arbitrary such that $t \to v$.
Since $u \to t$ and $t \to v$, by **Transitivity** $u \to v$.
So $u \in P$ is such that $u \to v$.
By the arbitrariness of v and the assumption $t \in P$. □

Questions

General Question

1. Axiomatize the minimal propositional logic with respect to this relational semantics.
2. What is the notion of bisimulation for this propositional language in this relational semantics?
3. What is the fragment of first-order language corresponding to this propositional language in this relational semantics?

Specific Question about Ortho-logic and Its Extensions

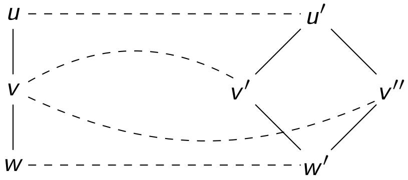
1. Is there a theory of modal companion for ortho-logic and its extensions?

Outline

Undefinability

Fact [Zhong, 2018a]

Separation is not modal definable.



- The left one is a bounded morphic image of the right one.
- The left one doesn't satisfy Separation, but the right one does.

Fact

Superposition is not modal definable.

Modal Logics

Theorem [Zhong, 2018b]

The modal logic **KTB** is sound and strongly complete with respect to the class of Kripke frames satisfying Reflexivity, Symmetry and Separation.

Theorem [Zhong, 2018b]

The following modal logic:

$$\mathbf{KTB} \oplus (\Box\Box p \rightarrow \Box\Box\Box p)$$

is sound and strongly complete with respect to the class of Kripke frames satisfying Reflexivity, Symmetry, Separation and Superposition.

An Important Validity

Proposition

The formula $\Box p \wedge \neg\Box q \to \Diamond(\Box p \wedge \Box\neg(\Box p \wedge \Box q))$ is valid in the class of all quantum Kripke frames.

Proof.

Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a model where $\mathfrak{F} = (\Sigma, \to)$ is a quantum Kripke frame.
For each $\phi \in \mathit{Form}$, let $[\![\phi]\!] \stackrel{\text{def}}{=} \{s \in \Sigma \mid \mathfrak{M}, s \Vdash \phi\}$.
Then $[\![\Box p]\!] = {\sim}(\Sigma \setminus [\![p]\!])$ and $[\![\Box q]\!] = {\sim}(\Sigma \setminus [\![q]\!])$.
Hence both of them are bi-orthogonally closed.
By orthomodularity $[\![\Box p]\!] \cap ({\sim}[\![\Box p]\!] \sqcup ([\![\Box p]\!] \cap [\![\Box q]\!])) \subseteq [\![\Box q]\!]$.
Hence $[\![\Box p]\!] \cap {\sim}([\![\Box p]\!] \cap {\sim}([\![\Box p]\!] \cap [\![\Box q]\!])) \subseteq [\![\Box q]\!]$.
Hence $[\![\Box p]\!] \cap [\![\Box\neg(\Box p \wedge \Box\neg(\Box p \wedge \Box q))]\!] \subseteq [\![\Box q]\!]$.
Hence $[\![\Box p \wedge (\Box\neg(\Box p \wedge \Box\neg(\Box p \wedge \Box q))) \to \Box q]\!] = \Sigma$.
Hence $\mathfrak{M} \Vdash \Box p \wedge (\Box\neg(\Box p \wedge \Box\neg(\Box p \wedge \Box q))) \to \Box q$.
Therefore, $\mathfrak{M} \Vdash \Box p \wedge \neg\Box q \to \neg\Box\neg(\Box p \wedge \Box\neg(\Box p \wedge \Box q))$. $\square$

A Modal Logic Sound w.r.t. Quantum Kripke Frames

Proposition

The following modal logic

$$\mathbf{KTB} \oplus \{\Box\Box p \to \Box\Box\Box p,\ \Box p \wedge \neg\Box q \to \Diamond(\Box p \wedge \Box\neg(\Box p \wedge \Box q))\}$$

is sound with respect to the class of all quantum Kripke frames

A Problem

Is there a special kind of Kripke frames which this modal logic axiomatizes?

Outline

How Is Quantum Theory Built?

1. Some data are obtained from experiments about microscopic objects.
2. They cannot be explained using classical physics.
3. Manipulate some complicated mathematical objects so that the outputs of the calculations fit the data.
4. von Neumann proposed the postulates of quantum theory, using Hilbert spaces.

Hilbert Space over $\mathbb{C}$

Definition (Hilbert Space over $\mathbb{C}$)

A Hilbert space over $\mathbb{C}$ is

1. a vector space over the complex numbers $\mathbb{C}$;
2. it is equipped with an inner product;
3. it is complete.

Hilbert Space and Quantum Theory

1. A quantum system is described by a Hilbert space $\mathcal{H}$ over $\mathbb{C}$.
2. The states of the system correspond to the one-dimensional subspaces of $\mathcal{H}$.
3. The properties of the system correspond to the subspaces of $\mathcal{H}$.

Quantum Logic

Aim: Rational Reconstruction of Quantum Theory

Paradigm:

1. Choose and start from physically transparent concepts.
2. Find simple and natural axioms to characterize the features of these concepts in quantum theory.
3. Use simple mathematical structures to model these concepts.
4. Prove representation theorems between these mathematical structures and Hilbert spaces.

Possible Benefits:

1. Highlight the quantum features of some basic physical concepts.
2. Understand the physical significance of the complicated structure of a Hilbert space.
3. Devise some (automatic) method for reasoning about quantum phenomena.
4. Popularize quantum theory in a simple but still rigorous way.

Approaches to Quantum Logic

1. Property - Algebraic Structure
   E.g. G. Birkhoff & J. von Neumann, *The Logic of Quantum Mechanics*, 1936
2. State - Relational Structure
   E.g. R. Goldblatt, *Semantic Analysis of Orthologic*, 1975
3. Composition of Systems - Category
   E.g. S. Abramsky & B. Coecke, *A Categorical Semantics of Quantum Protocols*, 2004

   ......

The Logic of Quantum Mechanics, 1936

The genesis of quantum logic is marked by the seminal paper

Birkhoff & von Neumann, *The Logic of Quantum Mechanics*, 1936

> *Our main conclusion ... is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to* set products, linear sums, *and* orthogonal complements-*and resembles the usual calculus of propositions with respect to* and, or, *and* not.
> [Birkhoff and von Neumann, 1936]

Hilbert Lattice

Fix a Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

Fact [Birkhoff and von Neumann, 1936], [Husimi, 1937]

The set $\mathcal{L}(\mathcal{H})$ of subspaces of $\mathcal{H}$ forms a *complete orthomodular lattice* $\mathfrak{L}(\mathcal{H})$:

- Partial Order: set-theoretic inclusion $\subseteq$;
- Meet: set-theoretic intersection $\cap$;
- Join: closure of the linear sum $\sqcup$;
- Top: $\mathcal{H}$;
- Bottom: $\{\mathbf{0}\}$;
- Orthocomplementation: orthocomplement $(\cdot)^{\perp}$.

Such a lattice is now called a Hilbert Lattice.

Orthomodular Lattice and Hilbert Lattice

Not every orthomodular lattice is a Hilbert lattice.

Theorem [Piron, 1976]

- The lattice of bi-orthogonally closed subspaces of a generalized Hilbert space is always a Piron lattice.
- Every Piron lattice of height at least 4 is isomorphic to the lattice of bi-orthogonally closed subsets of a generalized Hilbert space.

Key Lemma [Amemiya and Araki, 1966]

For every vector space V over $\mathbb{C}$ equipped with an inner product, the following are equivalent:

(i) it is metrically complete, and thus is a Hilbert space;

(ii) the bi-orthogonally closed subsets form an orthomodular lattice under $\subseteq$ and $(\cdot)^{\perp}$.

Beyond Piron's Result

- Piron's result shows a correspondence between Piron lattices and generalized Hilbert spaces.
- It's proved that there is a Piron lattice of infinite height that is not isomorphic to any Hilbert lattices. [Keller, 1980]
- Further conditions are required to characterize Hilbert lattices.
- Solèr shows that a generalized Hilbert space having an infinite 'orthonormal' sequence must be a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. [Solèr, 1995]
- Holland shows that this condition is equivalent to a lattice-theoretic condition. [Holland, 1995]
- To finally characterize Hilbert lattices, one need to distinguish among $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.
  The distinction is first-order in the language of fields, and is equivalent to first-order lattice-theoretic conditions.

Quantum Kripke Frame and Hilbert Space

Theorem [Zhong, 2015]

For each Kripke frame $\mathfrak{F} = (\Sigma, \to)$, the following are equivalent:

(i) it is a quantum Kripke frame, and there are $\{s_1, s_2, s_3, s_4\} \in \Sigma$ such that $s_i \not\to s_j$ for any distinct $i, j \in \{1, \ldots, 4\}$;

(ii) there are

1. a division ring $\mathbb{F}$ with involution;
2. a vector space V over $\mathbb{F}$ of dimension at least 4;
3. an orthomodular Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$;

such that $\mathfrak{F} \cong \mathfrak{F}_V$, where $\mathfrak{F}_V = (\Sigma(V), \to_V)$ is such that

1. $\Sigma(V)$ is the set of all one-dimensional subspaces of V;
2. for any $s, t \in \Sigma(V)$, $s \to_V t$, if $\langle \mathbf{u}, \mathbf{v} \rangle \neq 0$ for some $\mathbf{u} \in s$ and $\mathbf{v} \in t$.

Moreover, if they exist, both $\mathbb{F}$ and V are unique up to isomorphism, and $\langle \cdot, \cdot \rangle$ is unique up to a constant multiple.

Proof.

Use Piron's Theorem and the main result in [Zhong, 2017]. □

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Thank you very much!